Homework Set 0: Remembering Math

Due: Monday, September 2, 2024

To get your math neurons firing again, perform the following operations (YOU MAY USE THE CRC):

$$\frac{d}{dt}e^{-kt} = -ke^{-kt} \qquad \frac{d}{dt}\frac{1}{kt} = -\frac{1}{kt^2} \qquad \frac{d}{dt}\ln(kt) = \frac{k}{kt} = \frac{1}{t}$$

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$$\int e^{-kt}dt = \frac{-1}{k}e^{-kt} \qquad \int\frac{dt}{kt} = \frac{1}{k}\ln(kt) \qquad \int \ln(t)dt = t\ln(t) - t$$

#26 $$\int\frac{dt}{1+kt} = \frac{1}{k}\ln(1+kt) \qquad \left\{\frac{d}{dt}\left[\ln(1+kt)\right] = \frac{k}{1+kt}\right\}$$

#29 $$\int\frac{tdt}{1+kt} = \frac{1}{k}\int\frac{kt+1-1}{1+kt}dt = \frac{1}{k}\left[\int dt + \int\frac{dt}{1+kt}\right] = \frac{t}{k} - \frac{1}{k^2}\ln(1+kt)$$

$$\int\frac{tdt}{1+kt^2} = \frac{1}{2k}\int\frac{2kt\,dt}{1+kt^2} = \frac{1}{2k}\ln(1+kt^2)$$

The hyperbolic functions that we'll use are defined as

$$\sinh(z) = \frac{1}{2}\left(e^z - e^{-z}\right) \qquad \cosh(z) = \frac{1}{2}\left(e^z + e^{-z}\right)$$

$z = \sinh(w) = \frac{1}{2}\left(e^{w} - e^{-w}\right)$

Use the first one to show that (*"show" means start with the first expression and derive the second*)

$$\sinh^{-1}(z) = w \quad \Rightarrow \quad \sinh^{-1}(z) = \ln\left(z + \sqrt{z^2+1}\right)$$

(hint: start with $2z = e^{w} - e^{-w}$ and solve for w … you'll have to solve a quadratic in e^{2w} (& e^{w} & 1) and note that since the radical is always ±, technically, ±√ = +√)

$$\left(2z = e^{w} - e^{-w}\right)e^{w} \Rightarrow 2ze^{w} = e^{2w} - 1 \Rightarrow \underbrace{e^{2w} - 2ze^{w} - 1 = 0}_{\text{A QUADRATIC!}}$$

⇒ APPLY QUAD. FORMULA TO e^{w}!

$$\Rightarrow e^{w} = \frac{2z \pm \sqrt{4z^2 - 4(1)(-1)}}{2} = \frac{\cancel{2}z \pm \cancel{2}\sqrt{z+1}}{\cancel{2}}$$

$$\Rightarrow e^{w} = \underbrace{z \pm \sqrt{z+1}}_{} \Rightarrow \boxed{w = \ln\left(z + \sqrt{z+1}\right)}$$

SINCE SQUARE ROOTS ARE ALWAYS ±, WE CAN JUST WRITE $+\sqrt{\ }$

THUS! SINCE $\sinh^{-1}(z) = w$

$$\Rightarrow \boxed{\sinh^{-1}(z) = \ln\left(z + \sqrt{z+1}\right)}$$

Then show that

$$\frac{d}{dz}\ln\left(z + \sqrt{z^2+1}\right) = \frac{1}{\sqrt{z^2+1}}$$

$$\frac{1}{z + \sqrt{z^2+1}}\left[1 + \cancel{\tfrac{1}{2}}\left(z^2+1\right)^{-\frac{1}{2}}(\cancel{2}z)\right] =$$

$$= \left(\frac{1}{z+\sqrt{z^2+1}}\right)\left(1 + \frac{z}{\sqrt{z^2+1}}\right) = \left(\frac{1}{\cancel{z+\sqrt{z^2+1}}}\right)\left(\frac{\cancel{\sqrt{z^2+1} + z}}{\sqrt{z^2+1}}\right) = \boxed{\frac{1}{\sqrt{z^2+1}}} \text{ YEA!}$$

$$\Rightarrow \frac{d}{dz}\ln\left(z + \sqrt{z^2+1}\right) = \frac{1}{\sqrt{z^2+1}}$$

Then show that (THE CRC WILL GIVE YOU THE FIRST EXPRESSION. SHOW HOW TO GET THE SECOND.)

SHOWING THIS

QED!

$$\frac{d}{dz}\cosh^{-1}\left(e^{kz}\right) = \frac{k}{\sqrt{1 - e^{-2kz}}} = \frac{ke^{kz}}{\sqrt{e^{2kz} - 1}}$$

$$\left(\frac{k}{\sqrt{1 - e^{-2kz}}}\right)\left(\frac{e^{kz}}{e^{kz}}\right) = \frac{ke^{kz}}{\sqrt{\left(e^{2kz}\right)\left(1 - e^{-2kz}\right)}} = \boxed{\frac{ke^{kz}}{\sqrt{e^{2kz} - 1}}}$$